Conformal CR positive mass theorem

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Positive Mass Theorem

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$$g_{ij} = \delta_{ij} + O(|x|^{-\tau}),$$

$$|x||g_{ij,k}| + |x|^2|g_{ij,kl}| = O(|x|^{-\tau})$$

for some $\tau > (n-2)/2$. Here, $g_{ij,k}$ and $g_{ij,kl}$ are the covariant derivatives of g_{ij}.

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We also require

$$R_g = O(|x|^{-q})$$

for some $q > n$.

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The ADM mass of (M, g) is defined as

$$m_{ADM} = \frac{1}{4(n-1)\omega_{n-1}} \lim_{\Lambda \to \infty} \int_{\{|x|=\Lambda\}} \sum_{i,j=1}^{n} (g_{ij,i} - g_{ii,j})$$

Here, ω_{n-1} is the volume of the $(n-1)$-dimensional unit sphere.

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Example: $(\mathbb{R}^n, \delta)$ is asymptotically flat. The ADM mass of $(\mathbb{R}^n, \delta)$ is zero.

Positive Mass Theorem

Theorem (Positive Mass Theorem)

If (M, g) *is asymptotically flat with* $R_g \geq 0$*, then* $m_{ADM} \geq 0$ *and equality holds if and only if* $(M, g) \equiv (\mathbb{R}^n, \delta)$.

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Recently, Schoen-Yau claimed to prove the positive mass theorem in general.

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W. Simon (1999) proved the following:

Theorem (Conformal Positive Mass Theorem)

If $(M, \tilde{g})$ *and* (M, g) *are 3-dimensional asymptotically flat Riemannian manifolds with* $\tilde{g} = \phi^4 g$ *such that* $R_g - \phi^4 R_{\tilde{g}} \geq 0$,

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Taking $M = \mathbb{R}^3$ and $\tilde{g} = \delta$. Then we have:

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CR Positive Mass Theorem

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Let T be the unique vector field such that

$$\theta(T) = 1 \text{ and } d\theta(T, \cdot) = 0.$$

Also, let Z_1 be vector field such that

$$JZ_1 = iZ_1 \text{ and } JZ_{\overline{1}} = -iZ_{\overline{1}}.$$

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Let $(\theta, \theta^1, \theta^{\bar{1}})$ be dual to $(T, Z_1, Z_{\bar{1}})$ so that

$$d\theta = ih_{1\bar{1}}\theta^1 \wedge \theta^{\bar{1}}$$

with $h_{1\bar{1}} = 1$.

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The connection 1-form ω_1^1 and the torsion are determined by

$$d\theta^1 = \theta^1 \wedge \omega_1^1 + A^1_{\bar{1}}\theta \wedge \theta^{\bar{1}},$$
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The Tanaka-Webster curvature is given by

$$d\omega_1^1 = R\theta^1 \wedge \theta^{\bar{1}} (\mathrm{mod}\theta).$$

CR Positive Mass Theorem

Example: The Heisenberg group $\mathbb{H}^1 = \{(z,t) : z \in \mathbb{C}, t \in \mathbb{R}\}$, $J_0 : \mathbb{C} \to \mathbb{C}$ the standard complex structure, and

$$\overset{\circ}{\theta} = dt + iz d\bar{z} - i\bar{z} dz.$$

Then

$$\overset{\circ}{Z}_1 = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial z} + i\bar{z}\frac{\partial}{\partial t}\right), \overset{\circ}{Z}_{\bar{1}} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \bar{z}} - iz\frac{\partial}{\partial t}\right).$$

$$\overset{\circ}{\theta}{}^1 = \sqrt{2}dz, \overset{\circ}{\theta}{}^{\bar{1}} = \sqrt{2}d\bar{z}.$$

The Tanaka-Webster curvature $R = 0$.

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$$\theta = (1 + 4\pi A\rho^{-2} + O(\rho^{-3}))\overset{\circ}{\theta} + O(\rho^{-3})dz + O(\rho^{-3})d\bar{z},$$
$$\theta^1 = O(\rho^{-3})\overset{\circ}{\theta} + O(\rho^{-4})d\bar{z} + (1 + 2\pi A\rho^{-2} + O(\rho^{-3}))\sqrt{2}dz$$

for some constant A. Here,

$$\rho = \sqrt[4]{|z|^4 + t^2}.$$

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We also require that the Tanaka-Webster curvature $R \in L^1(N)$, i.e. $\int_N |R|\theta \wedge d\theta < \infty$.

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Example: The Heisenberg group $(\mathbb{H}^1, J_0, \overset{\circ}{\theta})$ is an asymptotically flat pseudohermitian manifold with *p*-mass $m(J_0, \overset{\circ}{\theta}) = 0$.

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Cheng-Malchiodi-Yang (2017) proved the following:

Theorem (CR Positive Mass Theorem)

If (N, J, θ) is a 3-dimensional asymptotically flat pseudohermitian manifold with $R \geq 0$ and the CR Paneitz operator is nonnegative, then its p-mass

$$m(J, \theta) \geq 0.$$

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Equality holds if and only if $(N, J, \theta) = (\mathbb{H}^1, J_0, \overset{\circ}{\theta})$.

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Fact: If the torsion $A_{11} = 0$, then the CR Paneitz operator is nonnegative.

Idea of the proof of CR Positive Mass Theorem:

Let $\beta : N \to \mathbb{C}$ be a smooth function such that

$$\beta = \bar{z} + \beta_{-1} + O(\rho^{-2+\epsilon}) \text{ near } \infty.$$

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Then one has the integral formula for the p-mass:

$$\begin{aligned}\frac{2}{3}m(J,\theta) = & -\int_N |\Box_b \beta|^2 \theta \wedge d\theta + 2\int_N |\beta_{,11}|^2 \theta \wedge d\theta \\ & + 2\int_N R|\beta_{,\bar{1}}|^2 \theta \wedge d\theta + \frac{1}{2}\int_N \bar{\beta} P \beta \theta \wedge d\theta.\end{aligned}$$

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Hsiao-Yung proved that there exists β such that $\Box_b \beta = 0$.

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Answer: Yes.

There exists some (N, J, θ) such that the CR Paneitz operator is not nonnegative, and its p-mass is negative.

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We have the following Conformal CR Positive Mass Theorem:

Theorem (H. 2017)

If $(N, J, \tilde{\theta})$ and (N, J, θ) are 3-dimensional asymptotically flat pseudohermitian manifolds with $\tilde{\theta} = \phi^2\theta$ such that $R - \phi^4\tilde{R} \geq 0$,

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Take $N = \mathbb{H}^1$ and $\tilde{\theta} = \overset{\circ}{\theta}$ in the above theorem. We have:

Theorem

If $(\mathbb{H}^1, \overset{\circ}{J}, \theta = \phi^{-2}\overset{\circ}{\theta})$ is an asymptotically flat pseudohermitian manifold such that $R \geq 0$, then $m(J, \theta) \geq 0$ and equality holds if and only if $\theta = \overset{\circ}{\theta}$.

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To solve the CR Yamabe problem, one tries to find the minimizer of the energy:

$$E(u) = \frac{\int_M \left((2+\frac{2}{n})|\nabla_b u|^2 + R_{\theta_0} u^2\right) dV_{\theta_0}}{\left(\int_M u^{2+\frac{2}{n}} dV_{\theta_0}\right)^{\frac{n}{n+1}}}$$

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Remark: Recall, there exists M such that the CR Paneitz operator is not nonnegative and its p-mass is negative. The minimizer may not exist on such M.

CR Yamabe flow

CR Yamabe flow is given by:

$$\frac{\partial}{\partial t}\theta(t) = -(R_{\theta(t)} - \overline{R}_{\theta(t)})\theta(t),$$

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- Zhang (2009) proved the long time existence and convergence when $Y(M, \theta_0) < 0$.
- When $Y(M, \theta_0) > 0$, Chang-Chiu-Wu (2010) proved the long time existence and convergence when $n = 1$ and torsion is zero.

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Suppose $M = \mathbb{S}^{2n+1}$. If $\theta(t)|_{t=0}$ is conformal to $\theta_{\mathbb{S}^{2n+1}}$, then CR Yamabe flow $\theta(t)$ converges to $\theta_{\mathbb{S}^{2n+1}}$.

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Using the CR positive mass theorem, we can prove:

Theorem (H.-Sheng-Wang 2017)

If (M, θ_0) is spherical or $dimM = 3$ such that the CR Paneitz operator is nonnegative, then CR Yamabe flow $\theta(t)$ converges.

Thank you very much for your attention!